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AN INTEGRAL RIEMANN-ROCH FORMULA FOR INDUCED REPRESENTATIONS OF FINITE GROUPS

BY

LEONARD EVENS AND DANIEL S. KAHN[1]

ABSTRACT. Let H be a subgroup of the finite group G, ξ a finite dimensional complex representation of H and ρ the induced representation of G. If $s_k(\rho) \in H^{2k}(G, \mathbf{Z})$, $k \geqslant 1$, denote the characteristic classes bearing the same relation to power sums that Chern classes bear to elementary symmetric functions, then we prove the following,

$$\bar{N}(k)\big(s_k(\rho) - \operatorname{Tr}_{H\to G}(s_k(\xi))\big) = 0,$$

where

$$\bar{N}(k) = \prod_{\substack{p \mid N(k) \\ p \text{ prime}}} p$$

and

$$N(k) = \Big(\prod_{p \text{ prime}} p^{[k/p-1]}\Big)\Big/k!.$$

(Tr denotes transfer.) Moreover, $\bar{N}(k)$ is the least integer with this property.

This settles a question originally raised in a paper of Knopfmacher in which it was conjectured that the required bound was $N(k)$.

1. Introduction. Let G be a finite group, and ρ: $G \to U_n$ a unitary representation of degree n. Various "characteristic classes" have been defined for ρ. The Chern classes $c_i(\rho) \in H^{2i}(G, \mathbf{Z})$, $0 \leqslant i \leqslant n$, were introduced by Atiyah in **[At]**. Other classes, $s_k(\rho) \in H^{2k}(G, \mathbf{Z})$, $1 \leqslant k < \infty$, which bear the same relation to power sums that Chern classes bear to elementary symmetric functions, were first studied in the context of group cohomology by Knopfmacher **[Kn]** and were also investigated by C. B. Thomas (**[T1]** and **[T2]**). Knopfmacher showed that for ξ: $H \to U_n$ a representation of a subgroup $H \leqslant G$, the class $s_k(\rho)$, where $\rho = \operatorname{ind}_{H\to G}\xi$ (the induced representation), satisfies

$$\text{(K)} \qquad N_k\big(s_k(\operatorname{ind}_{H\to G}\rho) - \operatorname{Tr}_{H\to G}(s_k(\rho))\big) = 0$$

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where N_k is a (large) integer depending only on k and not on the group, subgroup, or representation. Knopfmacher asserts "that heuristic considerations (suggested ... by J. F. Adams and based on Riemann-Roch theorems) make it conceivable that the least positive integer N_k with the property ... is the number

$$N(k) = (k!)^{-1} \prod_{\text{prime } p} p^{[k/p-1]}."$$

It is our purpose here to establish a modified form of this conjecture. Namely, define

$$\bar{N}(k) = \prod_{\substack{p \text{ prime} \\ p | N(k)}} p$$

so that $\bar{N}(k)$ contains exactly those primes occurring in $N(k)$ *but to the first power*. Then, we shall demonstrate that $\bar{N}(k)$ is the least positive integer N_k such that (K) holds for all finite G, H, and ξ.

Our approach is computational, and we must admit that the number $\bar{N}(k)$ does not arise naturally from the argument. The reader might also find enlightening earlier partial results on the problem in **[T1]** and **[T2]**. (After we completed this work, J. C. Becker brought to our attention the Princeton thesis of F. Roush [1971] in which the modified conjecture was also proved. Roush's methods are substantially independent of ours.)

Before proceeding, we should note that the classes $s_k(\rho)$ seem to contain relatively little information, at least compared to the Chern classes. In §4, we work out a few examples which seem to support our pessimism about the utility of these classes. (See **[Ch1]** and **[Ch2]** for what we feel are much stronger results about the *Chern classes* of induced representations. There is some overlap between **[Ch2]** and this paper.)

2. **The case of** $P \int T$. We recall a few simple facts about the classes $s_k(\rho)$, $k = 0, 1, 2, \ldots$. (In what follows, "group" generally means "finite group," but in some cases "compact Lie group" is an allowed meaning. "Representation" means "finite dimensional unitary representation.") A good background source is **[H]**.

(a) If ρ and ξ are representations of the group G, then, for each $k \geqslant 0$,

$$s_k(\rho \oplus \xi) = s_k(\rho) + s_k(\xi).$$

(b) If ρ is a representation of G and ϕ: $H \to G$ is a homomorphism, then, for each $k \geqslant 0$,

$$s_k(\rho \circ \phi) = \phi^*(s_k(\rho)).$$

(c) Let ι: $T^n \to U_n$ be the inclusion of the n-dimensional torus in U_n. Writing, as usual, $H^*(BT^n, \mathbf{Z}) = \mathbf{Z}[x_1, \ldots, x_n]$, we have

$$s_k(\iota) = x_1^k + x_2^k + \cdots + x_n^k.$$

Notice that, for our purposes, we may essentially use (b) and (c) to define $s_k(\rho)$. Namely,

(B) $H^*(BU_n, \mathbf{Z}) = \mathbf{Z}[C_1, \ldots, C_n] \to H^*(BT^n, \mathbf{Z}) = \mathbf{Z}[x_1, \ldots, x_n]$

is a monomorphism onto the subring of symmetric polynomials. ($C_i \mapsto \phi_i =$ the ith elementary symmetric function.) Since $x_1^k + \cdots + x_n^k$ is symmetric, it has a unique pre-image, $S_k \in H^{2k}(BU_n, \mathbf{Z})$. Then, because of (b), given ρ: $G \to U_n$,

(Df) $$s_k(\rho) = \rho^*(S_k).$$

The relations between Chern classes c_i and the classes s_k are given by the so-called Newton formulas. For our purposes, it will be useful to express these as follows. Write $p_k = x_1^k + \cdots + x_n^k$. The usual derivation, [**W**, Chapter II, §3, p. 38], adapted to a calculation in the ring of formal power series $\mathbf{Z}[[x_1, \ldots, x_n]]$ yields

(Nt1) $$\sum_{k=1}^{\infty} (-1)^{k+1} p_k = \frac{\phi'}{\phi}$$

where $\phi = 1 + \phi_1 + \phi_2 + \cdots + \phi_n = \prod_{i=1}^{n}(1 + x_i)$ and $\phi' = \phi_1 + 2\phi_2 + \cdots + n\phi_n$. Because of the monomorphism (B), and (Df) above, this translates, for ρ: $G \to U_n$ into the formula

(Nt2) $$\sum_{k=1}^{\infty} (-1)^{k+1} s_k(\rho) = \frac{c'(\rho)}{c(\rho)}$$

where $c(\rho) = 1 + c_1(\rho) + \cdots + c_n(\rho)$ and $c'(\rho) = c_1(\rho) + 2c_2(\rho) + \cdots + nc_n(\rho)$. (The calculation should take place in $\prod_{k=0}^{\infty} H^{2k}(BG, \mathbf{Z})$. It makes sense since $c(\rho) = 1 + c_1 + \cdots + c_n$ is invertible there.)

In what follows, we shall make free use of the results on *wreath products* in [**N**] and [**Ch1**]; also, we use the terminology and notation of those papers. If S is any subgroup of S_n, we may form the wreath product, $S \int T = S \cdot T^n$, and this is a subgroup of U_n. Since $H^*(BT, \mathbf{Z}) = \mathbf{Z}[x]$ is $\mathbf{Z}$-free, there is an isomorphism

(Nk) $$H^*(B(S \int T), \mathbf{Z}) \cong H^*(S, H^*(BT^n, \mathbf{Z}))$$

which we call the *Nakaoka* isomorphism. (Nk) is a ring isomorphism, and it is natural with respect to changes in the subgroup S (for restriction or transfer). (See [**Ch1**, §2, pp. 181–183] for details.) Below, we are interested in the special case $n = p$ is prime, and $S = P$ is cyclic of order p (say generated by the cycle $(123 \ldots p)$). Thus,

$$H^*(B(P \int T), \mathbf{Z}) \cong H^*(P, \mathbf{Z}[x_1, x_2, \ldots, x_p])$$

where P permutes the variables cyclically.

As a P-module, each homogeneous component of $\mathbf{Z}[x_1, \ldots, x_p]$ breaks up into a direct sum of submodules either (i) isomorphic to $\mathbf{Z}[P]$ or (ii) isomorphic to $\mathbf{Z}$. The second type are all of the form $\mathbf{Z}\phi_p^j$ where $\phi_p = x_1x_2 \ldots x_p$. In particular, for degrees $1, 2, \ldots, p-1$ (in the x_i), only type (i) submodules occur. Since $H^r(P, \mathbf{Z}[P]) = 0$, $r > 0$, we can prove

LEMMA. *The total Chern class of the inclusion* ι': $P \int T \to U_p$ *is given by*

$$c(\iota') = \phi - \mu^{p-1}$$

where μ *generates* $H^2(P, \mathbf{Z})$.

PROOF. In essence, this is proved in [**Ch1**, Theorem 4 and Corollary 5]. However, it follows easily from the above remark and identification of the components of the Chern classes on the "edges", $H^0(P, \mathbf{Z}[x_1, \ldots, x_n])$ and $H^*(P, \mathbf{Z})$. (See [**Ch1**, §3, p. 185].)

Note. See also [**Ch2**, Lemma in Theorem I] where the same lemma plays a role.

We shall now derive the crucial formula.

PROPOSITION I. Write $s_k = s_k(\iota')$. Then, with the notation as above,

$$\sum_{k=1}^{\infty} (-1)^{k+1}(s_k - p_k) = \frac{\mu^{p-1}}{1 - \mu^{p-1} + \phi_p} = \sum_{n=0}^{\infty} \mu^{p-1}\left(\mu^{p-1} - \phi_p\right)^n.$$

Note. For $k = 1, 2, \ldots, p-2$, the above formula yields $s_k = p_k$, and for $k = p-1, p$,

$$s_{p-1} = p_{p-1} - \mu^{p-1}, \quad s_p = p_p.$$

These latter results were obtained by Thomas in [**T1**], also by means of Newton's formulas.

PROOF OF PROPOSITION. By (Nt2), we have

$$\sum_{k=1}^{\infty} (-1)^{k+1} s_k = \frac{c'}{c}.$$

Also, on the "edge", $H^0(P, \mathbf{Z}[x_1, \ldots, x_n])$,

$$\sum_{k=1}^{\infty} (-1)^{k+1} p_k = \frac{\phi'}{\phi}.$$

By the lemma,

$$c = \phi - \mu^{p-1}, \quad c' = \phi' - (p-1)\mu^{p-1} = \phi' + \mu^{p-1}$$

since $p\mu = 0$. Thus,

$$\sum_{k=1}^{\infty}(-1)^{k+1}(s_k - p_k) = \frac{\phi' + \mu^{p-1}}{\phi - \mu^{p-1}} - \frac{\phi'}{\phi}$$
$$= \frac{\phi'\phi + \mu^{p-1}\phi - \phi\phi' + \phi'\mu^{p-1}}{\phi(\phi - \mu^{p-1})}$$
$$= \frac{\phi\mu^{p-1} + \phi'\mu^{p-1}}{\phi(\phi - \mu^{p-1})}.$$

(Note that all this makes sense since ϕ and $\phi - \mu^{p-1}$ are invertible in $\prod_{k=0}^{\infty} H^{2k}(B(P \int T), \mathbf{Z})$.)

To reduce this further, we must say something about the product structure in $H^*(P, \mathbf{Z}[x_1, \ldots, x_p])$. Given $\alpha \in H^r(P, \mathbf{Z})$, $r > 0$, and $\beta \in H^0(P, M) = M^P$, where M is a direct summand of $\mathbf{Z}[x_1, \ldots, x_p]$ (as described above), then $\alpha\beta$ may be identified with an element of $H^r(P, M)$, which is in turn a direct summand of $H^r(P, \mathbf{Z}[x_1, \ldots, x_p])$. In particular, if $M \cong \bigoplus \mathbf{Z}[P]$, then $\alpha\beta = 0$, whatever α may be. Thus, the following products are trivial.

Product	*P-Module*
$\mu^{p-1}\phi_j = 0, \quad j = 1, 2, \ldots, p-1$	$\bigoplus_{\substack{i_1, \ldots, i_j \\ \text{distinct}}} \mathbf{Z}x_{i_1}x_{i_2} \cdots x_{i_j} \cong \mathbf{Z}[P]^{\binom{p}{j}/p}$
$\mu^{p-1}p_k = 0, \quad k = 1, 2, \ldots$	$\mathbf{Z}x_1^k + \mathbf{Z}x_2^k + \cdots + \mathbf{Z}x_p^k \cong \mathbf{Z}[P]$.

By an analogous argument, $\mu^i\phi_p^j \neq 0$ since $M = \mathbf{Z}\phi_p^j \cong \mathbf{Z}$ as a P-module.

We may now calculate further.

$$\mu^{p-1}\phi' = \mu^{p-1}(\phi_1 + 2\phi_2 + \cdots + p\phi_p)$$
$$= p\mu^{p-1}\phi_p = 0.$$

Hence,

$$\sum_{k=1}^{\infty}(-1)^{k+1}(s_k - p_k) = \frac{\phi\mu^{p-1}}{\phi(\phi - \mu^{p-1})} = \frac{\mu^{p-1}}{\phi - \mu^{p-1}}.$$

Write $\phi = 1 + \dot{\phi}$. Then, the above quotient may be expanded

$$\frac{\mu^{p-1}}{1 - (\mu^{p-1} - \dot{\phi})} = \mu^{p-1}\left(1 + (\mu^{p-1} - \dot{\phi}) + (\mu^{p-1} - \dot{\phi})^2 + \ldots\right).$$

Notice, that in the product on the right, any terms involving $\phi_1, \phi_2, \ldots,$ or ϕ_{p-1} ultimately yield zero when multiplied by μ^{p-1}. Hence $\dot{\phi}$ may be replaced by ϕ_p, and we obtain, as required,

$$\frac{\mu^{p-1}}{1-\left(\mu^{p-1}-\phi_p\right)} = \mu^{p-1}\left(1+\mu^{p-1}-\phi_p+\left(\mu^{p-1}-\phi_p\right)^2+\dots\right).$$

Let $N(k) = \prod p^{[k/(p-1)]}/k!$ as before, and put $\delta_k = s_k - p_k$.

THEOREM II. *With the notation as above, in* $P \int T, p\delta_k = 0$ *in any event*; *and* $\delta_k \neq 0 \Leftrightarrow p|N(k)$.

Note. In essence, Theorem II tells us the exact order of the difference $s_k(\text{ind }\xi) - \text{tr}(s_k(\xi))$ for ξ: $T^p \to T = U_1$ the projection onto the first factor. It turns out to be the power (≤ 1) of p in $N(k)$.

PROOF OF THEOREM II. For k a natural number, set $k = p^{\nu_p(k)}k'$ where $(k', p) = 1$. Then one knows that

$$\nu_p(k!) = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots + \left[\frac{k}{p^d}\right] + \dots.$$

Hence,

$$\nu_p(N(k)) = \left[\frac{k}{p-1}\right] - \left[\frac{k}{p}\right] - \left[\frac{k}{p^2}\right] - \cdots - \left[\frac{k}{p^d}\right] - \dots.$$

Write

$$\begin{aligned} k &= k_1 p + r_0 \\ k_1 &= k_2 p + r_1 \\ &\vdots \\ k_d &= k_{d+1} p + r_d \\ &\vdots \end{aligned}$$

where $0 \leq r_j < p, j = 0, 1, 2, \dots,$ and note

$$k = r_0 + r_1 p + r_2 p^2 + \cdots + r_d p^d + \dots$$

is the p-adic expansion of k. Also, $k_1 = [k/p]$, $k_2 = [k_1/p] = [k/p^2]$, etc.

We can rewrite

$$\begin{aligned} k &= k_1(p-1) + k_1 + r_0 \\ k_1 &= k_2(p-1) + k_2 + r_1 \\ &\vdots \end{aligned}$$

Hence,

$$\left[\frac{k}{p-1}\right] = k_1 + k_2 + \cdots + k_d + \cdots + \left[\frac{r_0 + r_1 + r_2 + \dots}{p-1}\right].$$

Thus,

$$\nu_p(N(k)) = \left[\frac{r_0 + r_1 + \cdots + r_d + \dots}{p-1} \right],$$

and

(L1) $$\nu_p(N(k)) > 0 \Leftrightarrow r_0 + r_1 + \cdots + r_d + \dots \geqslant p - 1.$$

Our task now is to determine the nonzero terms in the expansion of Proposition I,

(F2) $$\sum_{k=1}^{\infty} (-1)^{k+1} \delta_k = \sum_{n=0}^{\infty} \mu^{p-1} \left(\mu^{p-1} - \phi_p \right)^n.$$

(In any event, $p\mu = 0$ so $p\delta_k = 0$ for all $k \geqslant 1$.)

Fix an n in (F2), and write

$$n = s_0 + s_1 p + s_2 p^2 + \cdots + s_d p^d + \dots$$

with $0 \leqslant s_j < p, j = 0, 1, 2, \dots.$

Then

(F3) $$\left(\mu^{p-1} - \phi_p \right)^n \equiv \left(\mu^{p-1} - \phi_p \right)^{s_0} \left(\mu^{(p-1)p} - \phi_p^p \right)^{s_1} \dots \left(\mu^{(p-1)p^d} - \phi_p^{p^d} \right)^{s_d} \dots.$$

The congruence is modulo p, but *after multiplication by* μ^{p-1}, it may be replaced by *equality*. Suppose

$$i = t_0 + t_1 p + t_2 p^2 + \cdots + t_d p^d + \dots$$

with $0 \leqslant t_j \leqslant s_j$, $j = 0, 1, 2, \dots.$ We shall abbreviate these coefficient inequalities by $i \leqslant_p n$. Then, for each $i \leqslant_p n$, the term

$$\pm h_{n,i} \mu^{i(p-1)} \phi_p^{n-i}$$

occurs on the right where

$$h_{n,i} = \binom{s_0}{t_0} \binom{s_1}{t_1} \cdots \binom{s_d}{t_d} \cdots \not\equiv 0 \bmod p.$$

Moreover, *these are the only terms which occur there*. Thus, the expression on the right of (F2) is a sum of *nonzero* terms of the form

(F4) $$\pm h_{n,i} \mu^{p-1} \mu^{i(p-1)} \phi_p^{n-i} \quad \text{with } i \leqslant_p n.$$

Looking at the degrees in μ and ϕ_p separately, we see that these terms form a linearly independent set. Moreover, the total degree (divided by 2) in (F4) is $p - 1 + i(p-1) + (n-i)p = p - 1 + np - i$. Thus,

(L2) $\delta_k \neq 0 \Leftrightarrow$ There is a representation $k = p - 1 + np - i$ with $n, i \geqslant 0$ and $i \leqslant_p n$.

Thus, we need to show the equivalence of the conditions on the right of (L1) and (L2).

LEMMA 3. *With the notation as above, if* $k = p - 1 + np - i$ *with* $n, i \geqslant 0$ *and* $i \leqslant_p n$, *then* $r_0 + r_1 + \cdots + r_d + \ldots \geqslant p - 1$.

PROOF.

$$\begin{aligned} k &= p - 1 + (s_0 + s_1 p + \ldots)p - t_0 - t_1 p - \ldots \\ &= (p - 1 - t_0) + (\varepsilon_1 p + s_0 - t_1)p + (\varepsilon_2 p + s_1 - t_2 - \varepsilon_1)p^2 \\ &\quad + \cdots + (\varepsilon_d p + s_{d-1} - t_d - \varepsilon_{d-1})p^d + \ldots \end{aligned}$$

where $\varepsilon_0 = 0$ and inductively

$$\varepsilon_d = \begin{cases} 0 & \text{if } s_{d-1} - t_d - \varepsilon_{d-1} \geqslant 0, \\ 1 & \text{if } s_{d-1} - t_d - \varepsilon_{d-1} < 0. \end{cases}$$

Note, in any event, $-p \leqslant s_{d-1} - t_d - \varepsilon_{d-1} \leqslant p - 1$, so that

$$0 \leqslant \varepsilon_d p + s_{d-1} - t_d - \varepsilon_{d-1} < p.$$

Thus, we have determined the p-adic expansion of k.

$$\begin{aligned} r_0 &= p - 1 - t_0 \\ r_1 &= \varepsilon_1 p + s_0 - t_1 \\ r_2 &= \varepsilon_2 p + s_1 - t_2 - \varepsilon_1 \\ &\vdots \\ r_d &= \varepsilon_d p + s_{d-1} - t_d - \varepsilon_{d-1} \\ &\vdots \end{aligned}$$

Hence,

$$\begin{aligned} r_0 + r_1 + \cdots + r_d + \ldots &= p - 1 + (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d + \ldots)(p - 1) \\ &\quad + s_0 - t_0 + s_1 - t_1 + \cdots + s_d - t_d + \ldots. \end{aligned}$$

However,

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d + \ldots \geqslant 0$$

and, since $i \leqslant_p n$, by definition,

$$s_0 - t_0 + s_1 - t_1 + \cdots + s_d - t_d + \ldots \geqslant 0.$$

Thus, $r_0 + r_1 + \ldots \geqslant p - 1$ as required.

LEMMA 4. *With the notation as above, if* $r_0 + r_1 + \ldots \geqslant p - 1$, *then* $k = p - 1 + np - i$ *with* $n, i \geqslant 0$ *and* $i \leqslant_p n$.

PROOF.

$$k = r_0 + r_1 p + r_2 p^2 + \dots$$
$$= p - 1 + (r_1 + r_2 p + \dots)p - (p - 1 - r_0).$$

Try $i = t_0 = p - 1 - r_0$, $n = r_1 + r_2 p + \dots = k_1$. If $i \leqslant_p n$, we are done. But,

$$t_0 \leqslant r_1 \Leftrightarrow p - 1 - r_0 \leqslant r_1 \Leftrightarrow p - 1 \leqslant r_0 + r_1.$$

If $t_0 > r_1$, rewrite

$$k = p - 1 + (t_0 + r_2 p + \dots)p - (t_0 + (t_0 - r_1)p)$$
$$= p - 1 + (t_0 + r_2 p + \dots)p - (t_0 + t_1 p)$$

where $0 < t_1 = t_0 - r_1 = p - 1 - r_0 - r_1$ $(< p)$. Try $i = t_0 + t_1 p$, $n = t_0 + r_2 p + \dots$. Then

$$i \leqslant_p n \Leftrightarrow t_1 \leqslant r_2 \Leftrightarrow p - 1 - r_0 - r_1 \leqslant r_2 \Leftrightarrow p - 1 \leqslant r_0 + r_1 + r_2.$$

Continue in this way. Let d be the first index such that

$$r_0 + r_1 + r_2 + \cdots + r_d \geqslant p - 1.$$

Define

$$t_j = t_{j-1} - r_j = p - 1 - r_0 - r_1 - \cdots - r_j > 0$$

for $j = 0, 1, \dots, d - 1$. Then

$$k = p - 1 + \left(t_0 + t_1 p + \cdots + r_d p^{d-1} + \dots\right)p$$
$$- \left(t_0 + t_1 p + \cdots + t_{d-1} p^{d-1}\right),$$

and we can take

$$i = t_0 + t_1 p + \cdots + t_{d-1} p^{d-1},$$
$$n = t_0 + t_1 p + \cdots + r_d p^{d-1} + \dots .$$

Since $t_{d-1} = p - 1 - r_0 - r_1 - \cdots - r_{d-1} \leqslant r_d$, we are done.

3. **The general case.** We shall show that $\overline{N}(k) = \Pi_{p|N(k)} p$ is the optimal bound. First, we exhibit a class of groups for which $\overline{N}(k)$ is necessary.

For p a prime, let $\lambda_p\colon P \to T$ be a nontrivial 1-dimensional representation, and consider the p-dimensional representation of $G_p = P \int P$ given by $\rho_p = \iota' \circ (\mathrm{id} \int \lambda_p)$:

$$P \int P \xrightarrow{\mathrm{id}\int\lambda_p} P \int T \xrightarrow{\iota'} U_p.$$

Morever, let $\xi_p = \iota \circ (\lambda_p \times 1 \times \cdots \times 1)$:

$$H_p = P^p \xrightarrow{\lambda_p \times 1 \times \cdots \times 1} T^p \to U_p.$$

Then $\rho_p = \operatorname{ind} \xi_p$. Note $s_k(\xi_p) = s_k(\lambda_p) \times 1 \times \cdots \times 1$ where λ_p is viewed in $H^2(P, \mathbf{Z}) \cong \operatorname{Hom}(P, T)$. Also, $s_k(\lambda_p) = \lambda_p^k$.

We have a commutative diagram

$$\begin{array}{ccc} G_p & \to & P \int T \\ \uparrow & & \uparrow \\ H_p & \to & T^p \end{array}$$

which yields a transfer diagram

$$\begin{array}{ccc} H^{2k}(G_p, \mathbf{Z}) & \leftarrow & H^{2k}(B(P \int T), \mathbf{Z}) \\ \uparrow \mathrm{Tr} & & \uparrow \mathrm{Tr} \\ H^{2k}(H_p, \mathbf{Z}) & \leftarrow & H^{2k}(B(T^p), \mathbf{Z}) \end{array}$$

On the other hand, a simple calculation shows

$$\begin{aligned} p_k &= x_1^k + \cdots + x_p^k \\ &= \mathrm{Tr}_{T^p \to P \int T}(x_1^k) \in H^0(P, H^{2k}(BT^p)) \subset H^{2k}(B(P \int T)), \end{aligned}$$

where $x_1^k \in H^{2k}(BT^p, \mathbf{Z})$ = component of $\mathbf{Z}[x_1, \ldots, x_p]$ of degree k. (See [**Ch1**, §2, pp. 181–182]. We make use of the Nakaoka isomorphism and its naturality with respect to transfer.)

Hence,

$$\begin{aligned} s_k(\rho_p) &= (\mathrm{id} \int \lambda_p)^*(s_k) = (\mathrm{id} \int \lambda_p)^*(p_k + \delta_k) \\ &= (\mathrm{id} \int \lambda_p)^*(\mathrm{Tr}\, x_1^k) + (\mathrm{id} \int \lambda_p)^*(\delta_k) \\ &= \mathrm{Tr}(\lambda_p \times 1 \times \cdots \times 1)^*(x_1^k) + \Delta_{k,p} \\ &= \mathrm{Tr}(\lambda_p^*(x_1)^k \times 1 \times \cdots \times 1) + \Delta_{k,p} \end{aligned}$$

where $\Delta_{k,p} = (\mathrm{id} \int \lambda_p)^*(\delta_k)$. Thus,

$$s_k(\rho_p) - \mathrm{Tr}(s_k(\xi_p)) = \Delta_{k,p}.$$

We shall show

(I1) $$\Delta_{k,p} \neq 0 \Leftrightarrow \delta_k \neq 0 \Leftrightarrow p | N(k).$$

To see this, note first that $\delta_k \in H^*(B(P \int T), \mathbf{Z})$. Also, if $\bar{\mu} \in H^2(P, \mathbf{Z}/p\mathbf{Z})$ is the reduction mod p of μ, then the homomorphism

$$\mu H^*(B(P \int T), \mathbf{Z}) \to \bar{\mu} H^*(B(P \int T), \mathbf{Z}/p\mathbf{Z})$$

is a monomorphism. (For, we may use the Nakaoka isomorphisms and note that $H^{2r}(P, \mathbf{Z}) \to H^{2r}(P, \mathbf{Z}/p\mathbf{Z})$, $r > 0$, is an isomorphism. Since $H^r(P, \mathbf{Z}[x_1, \ldots, x_p]) \cong \bigoplus H^r(P, \mathbf{Z})$ for $r > 0$ by the P-module argument of §2, the connection is clear.) Moreover, for coefficients in $\mathbf{Z}/p\mathbf{Z}$, the Nakaoka argument works also for $P \int P$, and we have a commutative diagram

$$\begin{array}{ccc} H^*(P \int P, \mathbf{Z}/p\mathbf{Z}) & \leftarrow & H^*(B(P \int T), \mathbf{Z}/p\mathbf{Z}) \\ \wr\Vert & & \wr\Vert \\ H^*(P, \mathbf{Z}/p\mathbf{Z}[u_1, u_2, \ldots, u_p, \beta u_1, \ldots, \beta u_p]) & \leftarrow & H^*(P, \mathbf{Z}/p\mathbf{Z}[\bar{x}_1, \ldots, \bar{x}_p]) \end{array}$$

where $u_1, \ldots, u_p$ form a basis for $H^1(P^p, \mathbf{Z}/p\mathbf{Z})$ generating an exterior algebra, and their Bocksteins $\beta u_1, \ldots, \beta u_p$ generate a polynomial subring. The lower arrow is induced from $\bar{x}_i \to \beta u_i$, $i = 1, 2, \ldots, p$.

Since $\mathbf{Z}/p\mathbf{Z}[\bar{x}_1, \ldots, \bar{x}_p] \cong \mathbf{Z}/p\mathbf{Z}[\beta u_1, \ldots, \beta u_p]$ is a P-direct summand of $H^*(P^p, \mathbf{Z}/p\mathbf{Z})$, it follows that the lower (and hence upper) arrow is a monomorphism. That $\Delta_{k,p} \neq 0 \Leftrightarrow \delta_k \neq 0$ follows from the commutative diagram:

$$\begin{array}{lccc}
\Delta_{k,p} & H^*(P \int P, \mathbf{Z}) & \longleftarrow & H^*(B(P \int T), \mathbf{Z}) & \delta_k \\
& \downarrow & & \downarrow \text{mono on ideal generated by } \mu & \\
\bar{\Delta}_{k,p} & H^*(P \int P, \mathbf{Z}/p\mathbf{Z}) & \underset{\text{mono}}{\longleftarrow} & H^*(B(P \int T), \mathbf{Z}/p\mathbf{Z}) & \bar{\delta}_k
\end{array}$$

Note. If we write $\omega_p = (\mathrm{id} \int \lambda_p)^*(\phi_p)$, then we have the formula

(D1) $$\sum_{k=1}^{\infty} (-1)^{k+1} \Delta_{k,p} = \frac{\mu^{p-1}}{1 - \mu^{p-1} + \omega_p}.$$

(ω_p is just the wreath product class: $\omega_p = 1 \int \lambda_p$. See [Ch1, Theorem 3] and [N, §4].) Similarly, if we take $\lambda_p = 1$ instead, then $\mathrm{id} \int 1 = \pi_p$, $\mathrm{Tr}(s_k(1)) = 0$ for $k > 0$, and we obtain the formula

(D2) $$\sum_{k=1}^{\infty} (-1)^{k+1} s_k(\pi_p) = \frac{\mu^{p-1}}{1 - \mu^{p-1}}.$$

Thus,

$$s_k(\pi_p) = \begin{cases} 0, & k \not\equiv 0 \bmod p - 1, \\ \pm \mu^k, & k \equiv 0 \bmod p - 1. \end{cases}$$

We shall use these facts later.

On the basis of (I1), we can conclude that the required bound is no smaller than $\bar{N}(k)$. For each prime p dividing $\bar{N}(k)$, there is a group $P \int P$ and induced representation ρ_p such that $s_k(\rho_p)$ is of order p. (However, one can construct a single group G_k, and an induced representation ρ_k for which $s_k(\bar{\rho}_k)$ is of order $\bar{N}(k)$ so that all primes are dealt with simultaneously by the same group. To do so requires an extended discussion which is not essential for our basic argument.)

We now show that $\bar{N}(k)$ is a *universal* bound for all groups.

THEOREM III. *Fix an integer $k > 0$, and let $\bar{N}(k)$ be the product of all distinct primes p which divide $N(k) = (\prod_p p^{[k/(p-1)]})/k!$. Then for every finite* *group G, subgroup H, and finite dimensional unitary representation ξ of H,*

$$\bar{N}(k)\left[s_k(\mathrm{ind}_{H\to G}\xi) - \mathrm{Tr}_{H\to G}(s_k(\xi))\right] = 0.$$

Moreover, $\bar{N}(k)$ is the least natural number with this property.

PROOF. We have just dealt with the second assertion. For the first assertion, we proceed as follows.

First, we reduce to the case of p-groups. To do so, consider general $G > H$ and ξ. For each prime p, let Q be a p-Sylow subgroup of G. For any $s \in H^*(H, \mathbf{Z})$, we have the double coset rule [CE, Chapter XII, Proposition 9.1].

$$\text{(DCT)} \qquad \mathrm{res}_{G\to Q}(\mathrm{tr}_{H\to G}(s)) = \sum_{i=1}^{d} \mathrm{tr}_{H_i\cap Q\to Q}\left(\mathrm{res}_{H_i\to H_i\cap Q}(c_i(s))\right)$$

where $G = \bigcup_{i=1}^{d} Qg_iH$ is a double coset decomposition of G, $H_i = g_iHg_i^{-1}$, and c_i: $H^*(H, \mathbf{Z}) \to H^*(H_i, \mathbf{Z})$ is induced by the homomorphism $H_i \to H$ given by $h' \to g_i^{-1}h'g_i$. Similarly, for ξ a representation of H, the original double coset rule for induced representations reads

$$\text{(DCM)} \qquad \mathrm{res}_{G\to Q}(\mathrm{ind}_{H\to G}\xi) = \sum_{i=1}^{d} \mathrm{ind}_{H_i\cap Q\to Q}\left(\mathrm{res}_{H_i\to H_i\cap Q}c_i(\xi)\right)$$

(as above, $c_i(\xi)(h') = \xi(g_i^{-1}h'g_i)$).

Since restriction from G to Q is a *monomorphism* on the p-primary component, to prove the theorem, it suffices to prove (for $k > 0$)

$$\text{(P)} \qquad \bar{N}(k)\left[\mathrm{res}_{G\to Q}(s_k(\mathrm{ind}_{H\to G}\xi)) - \mathrm{Tr}_{H\to G}(s_k(\xi))\right] = 0$$

for every prime p dividing the order of G. Using the two double coset rules (and the properties (a) and (b) of s_k), we can reduce the expression in *brackets* in (P) to

$$\text{(Q)} \qquad \sum_{i=1}^{d}\left[s_k(\mathrm{ind}_{H_i\cap Q\to Q}\xi_i) - \mathrm{Tr}_{H_i\cap Q\to Q}(s_k(\xi_i))\right]$$

where we abbreviate $\xi_i = \mathrm{res}_{H_i\to H_i\cap Q}(c_i(\xi))$. If we establish the theorem for p-groups, then $\bar{N}(k)$ works for each term of the sum and hence for the sum.

We now suppose that G is a p-group. Because of transitivity, we may further assume that H is a (necessarily normal) subgroup of index p. For, let $G > G_1 > H$, and suppose the theorem has been established for index $< (G$:

H). Then,

$$\begin{aligned}
&\bar N(k)\left[s_k(\mathrm{ind}_{H\to G}\xi) - \mathrm{Tr}_{H\to G}(s_k(\xi))\right] \\
&\quad = \bar N(k)\left[s_k\left(\mathrm{ind}_{G_1\to G}(\mathrm{ind}_{H\to G_1}\xi)\right) - \mathrm{Tr}_{G_1\to G}\left(s_k(\mathrm{ind}_{H\to G_1}\xi)\right)\right. \\
&\qquad\quad \left. + \mathrm{Tr}_{G_1\to G}\left(s_k(\mathrm{ind}_{H\to G_1}\xi)\right) - \mathrm{Tr}_{G_1\to G}\left(\mathrm{Tr}_{H\to G_1}(s_k(\xi))\right)\right] \\
&\quad = \bar N(k)\left[s_k(\mathrm{ind}_{G_1\to G}\xi_1) - \mathrm{Tr}_{G_1\to G}s_k(\xi_1)\right] \\
&\qquad\quad + \mathrm{Tr}_{G_1\to G}\left(\bar N(k)\left[s_k(\mathrm{ind}_{H\to G_1}\xi) - \mathrm{Tr}_{H\to G_1}(s_k(\xi))\right]\right).
\end{aligned}$$

(We have abbreviated $\xi_1 = \mathrm{ind}_{H\to G}\xi$.) However, both terms in this expression have been inductively assumed to be zero.

We can now suppose that H is a normal subgroup of G of index p. If so, there is a factorization of the induced representation, $\rho = \mathrm{ind}_{H\to G}\xi$,

$$G \xrightarrow{\Phi} P \int H \xrightarrow{\mathrm{id}\ \xi} P \int U_n \xrightarrow{\bar\iota} U_{pn}, \qquad \rho: G \to U_{pn}$$

where $n = \deg \xi$. (Φ is defined in [**N**, §2, pp. 54–55]; see also [**Ch1**, p. 189] and [**Ch2**].) Hence, we must analyze the class $s_k(\iota) \in H^{2k}(B(P \int U_n), \mathbf{Z})$. To this end, recall the Nakaoka isomorphism (§2)

$$H^*(B(P \int U_n), \mathbf{Z}) \cong H^*(P, H^*(BU_n, \mathbf{Z})^p).$$

Let S_k be the *universal* class in $H^{2k}(BU_n, \mathbf{Z})$, and write

$$s_{k,i} = 1 \times 1 \times \cdots \times \underset{\substack{\uparrow \\ i\text{th position}}}{S_k} \times \cdots \times 1.$$

Then

$$\mathrm{tr}(s_{k,1}) = s_{k,1} + s_{k,2} + \cdots + s_{k,p} \in H^{2k}\left((BU_n)^p, \mathbf{Z}\right)^P.$$

Lemma. *With the notation as above, for* $k > 0$,

$$\bar N(k)\left[s_k(\bar\iota) - \mathrm{tr}(s_{k,1})\right] = 0.$$

Proof of lemma. One knows that

$$H^*(BU_n, \mathbf{Z}) \to H^*(BT^n, \mathbf{Z}) = \mathbf{Z}[x_1, x_2, \ldots, x_n]$$

is a monomorphism onto the *direct summand* of symmetric polynomials. (See [**Bor**, §20, p. 66].) Hence,

$$H^*(BU_n, \mathbf{Z})^p \to H^*(BT^n, \mathbf{Z})^p$$

is P-monomorphism onto a P-direct summand. Hence, the restriction

$$\begin{aligned}
H^*(B(P \int U_n)) &\cong H^*(P, H^*(BU_n)^p) \\
&\to H^*(P, H^*(BT^n)^p) \cong H^*(B(P \int T^n))
\end{aligned}$$

is also a monomorphism, and it suffices to prove an analogous result on the right. In more detail, let $P^* \cong P$ be the subgroup of $\mathfrak{S}_{np}$ which permutes the p blocks $\{1, 2, \ldots, n\}$, $\{n+1, \ldots, 2n\}, \ldots, \{(p-1)n+1, \ldots, pn\}$ cyclically. Then,

$$P \int_p (T^n) \cong P \int_p (1 \int_n T) \cong (P \int_p 1) \int_{pn} T = P^* \int_{pn} T.$$

Writing

$$\begin{aligned} H^*(BT^{pn}) &= H^*(BT^n)^p \\ &\cong \mathbf{Z}[x_{11}, \ldots, x_{1n}] \otimes \mathbf{Z}[x_{21}, \ldots, x_{2n}] \otimes \cdots \otimes \mathbf{Z}[x_{p1}, \ldots, x_{pn}] \\ &\cong \mathbf{Z}[\{x_{ij} | i = 1, \ldots, p; j = 1, \ldots, n\}], \end{aligned}$$

$p_{k,i} = x_{i1}^k + x_{i2}^k + \cdots + x_{in}^k$, we note that $s_{k,i} \to p_{k,i}$, and

$$\operatorname{tr}(p_{k,1}) = p_{k,1} + p_{k,2} + \cdots + p_{k,p} = \sum_{i,j} x_{ij}^k.$$

(Denote this last sum simply p_k'.) Thus, if ι': $P^* \int_{pn} T \to U_{pn}$ is the inclusion, we need to show

$$\text{(K1)} \qquad \bar{N}(k)[s_k(\iota') - p_k'] = 0.$$

(Transfer commutes with restriction in this case since

$$(P \int_p U_n : U_n^p) = (P \int_p T^n : (T^n)^p) = p.)$$

To establish (K1), it suffices to work in any *larger* wreath product $S \int_{pn} T$ where $P < S < \mathfrak{S}_{np}$. If we do so, $p_k' = \sum_{i,j} x_{ij}^k \in H^{2k}(BT^{np}, \mathbf{Z})^S$ need not be changed, but ι' is replaced by the inclusion ι: $S \int_{pn} T \to U_{pn}$.

The appropriate group to use is $S = P^n$ where the jth factor P of the product permutes the jth *column* $\{x_{ij} | i = 1, 2, \ldots, p\}$ of the matrix of indeterminates. P^* is imbedded diagonally in P^n. For this choice,

$$P^n \int_{pn} T \cong (P \int_p T)^n$$

and

$$\iota'' \cong \iota \times \iota \times \cdots \times \iota \quad (n \text{ times})$$

where ι: $P \int T \to U_p$ is the inclusion studied in §2. Moreover,

$$\text{(K2)} \qquad \begin{aligned} s_k(\iota \times \iota \times \cdots \times \iota) = s_k(\iota) \times 1 \times \cdots \times 1 + 1 \times s_k(\iota) \times \cdots \times 1 \\ + \cdots + 1 \times 1 \times \cdots \times s_k(\iota). \end{aligned}$$

On the other hand, p_k' may be rewritten

$$\text{(K3)} \qquad \begin{aligned} p_k' = \sum_{j=1}^{n} \sum_{i=1}^{p} x_{ij}^k = p_k \times 1 \times \cdots \times 1 + 1 \times p_k \times \cdots \times 1 \\ + \ldots + 1 \times 1 \times \cdots \times p_k \end{aligned}$$

where (as in §2) $p_k = x_{11}^k + x_{21}^k + \cdots + x_{p1}^k$, and we make suitable identifications involving cross products. Subtract (K3) from (K2) and recall from Theorem II that $\bar{N}(k)[s_k(\iota) - p_k] = 0$. That completes the proof of the lemma.

Note. Subtler versions of the above arguments are used in **[Ch2]** to calculate the Chern classes of ι.

To prove the theorem, consider the diagram

$$\text{(K4)}\qquad \begin{array}{ccccc} G & \overset{\Phi}{\to} & P \wr H & \overset{\mathrm{id} \wr \xi}{\to} & P \wr U_n \\ \uparrow & & \uparrow & & \uparrow \\ H & \overset{\Phi'}{\to} & H^p & \overset{\xi^p}{\to} & U_n^p \end{array}$$

where Φ' is the restriction of Φ to the normal subgroup H. In fact,

$$\Phi'(h) = h \times g_2^{-1} h g_2 \times \cdots \times g_p^{-1} h g_p$$

where $g_1 = 1, g_2, \ldots, g_p$ forms a set of left coset representatives of H in G. (See [N, §2].) The subgroups in the diagram (K4) are each of index p, and there results a commutative transfer diagram. Thus,

$$\begin{aligned} (\mathrm{id} \wr \xi)^*(\mathrm{tr}(s_{k,1})) &= \mathrm{tr}((\xi^p)^*(s_{k,1})) \\ &= \mathrm{tr}((\xi \times \cdots \times \xi)^*(S_k \times 1 \times \cdots \times 1)) \\ &= \mathrm{tr}(\xi^*(S_k) \times 1 \times \cdots \times 1) \\ &= \mathrm{tr}(s_k(\xi) \times 1 \times \cdots \times 1). \end{aligned}$$

Hence

$$\begin{aligned} \Phi^*\big((\mathrm{id} \wr \xi)^*(\mathrm{tr}(s_{k,1}))\big) &= \Phi^*\big(\mathrm{tr}(s_k(\xi) \times \cdots \times 1)\big) \\ &= \mathrm{tr}\big(\Phi'^*(s_k(\xi) \times 1 \times \cdots \times 1)\big) \\ &= \mathrm{tr}\big(s_k(\xi)\big). \end{aligned}$$

Since $\Phi^*((\mathrm{id} \wr \xi)^*(s_k(\bar{\iota}))) = s_k(\rho)$, we have

$$s_k(\rho) - \mathrm{tr}(s_k(\xi)) = \Phi^*\big((\mathrm{id} \wr \xi)^*[s_k(\bar{\iota}) - \mathrm{tr}(s_{k,1})]\big).$$

The theorem now follows from the lemma.

4. Some examples. We show in several examples that the classes $s_k(\rho)$ contain less information, in some sense, than the Chern classes.

(a) The *regular representation*. Let G be a finite group and ρ_G its regular representation. Then, $\rho = \mathrm{ind}_{\{1\} \to G} 1$, and since $\mathrm{Tr}(s_k(1)) = \mathrm{Tr}(0) = 0$ for $k > 0$, we conclude from Theorem III

$$\text{(E1)}\qquad \bar{N}(k) s_k(\rho_G) = 0, \qquad k > 0.$$

In particular if G is a p-group, then $p s_k(\rho_G) = 0$ for $k > 0$.

One knows, on the other hand, from an argument of Venkov [**Q**, §2, Theorem 2.1], that $H^*(G, \mathbf{Z})$ is a finite module over the subring $\mathbf{Z}[c_1(\rho_G), c_2(\rho_G), \ldots, c_g(\rho_G)]$ generated by the Chern classes. It follows that the exponent of $H^r(G, \mathbf{Z})$, for r sufficiently large, cannot be larger than the largest order of any of the Chern classes. Hence, the Chern classes of the regular representation generally have large orders while the orders of the classes $s_k(\rho_G)$ are small.

Note. To emphasize this point more concretely, we give the answers for a cyclic group G of order p^m (without proofs). χ denotes a generator of $H^2(G, \mathbf{Z})$.

(i) p *odd*,

$$c(\rho_G) = (1 - \chi^{p-1})^{p^{m-1}},$$

hence,

$$c_{p^m - p^{m-1}}(\rho_G) = -\chi^{p^m - p^{m-1}};$$

but

$$s_k(\rho_G) = \begin{cases} p^{m-1}\chi^k, & k \equiv 0 \bmod p - 1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $p = 2$;

$$c(\rho_G) = \left[(1 + \chi)(1 + 2\chi)(1 + 3\chi)\right]^{2^{m-2}},$$

but

$$s_k(\rho_G) = \begin{cases} 2^{m-1}\chi^k, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

(b) *Cyclic groups*.

PROPOSITION IV. *Let G be cyclic of order p^m, $m \geq 2$, H a subgroup of index p, $\lambda \in H^2(G, \mathbf{Z})$, and $\xi = \operatorname{res}_{G \to H} \lambda$. Then, for p odd,*

$$s_k(\operatorname{ind} \xi) = p\lambda^k = \operatorname{tr}(s_k(\xi)),$$

and for $p = 2$,

$$\begin{aligned} s_k(\operatorname{ind} \xi) &= 2\lambda^k = \operatorname{tr}(s_k(\xi)), \quad k \text{ even}, \\ &= 2\lambda^k + 2^{m-1}\lambda^k = \operatorname{tr}(s_k(\xi)) + 2^{m-1}\lambda^k, \quad k \text{ odd}. \end{aligned}$$

PROOF. Use Frobenius reciprocity and calculate.

COROLLARY V. *Let G be cyclic or order p^m, $m \geq 2$, H any nontrivial subgroup, and ξ a representation of H. Then, for p odd, or $p = 2$ and k even,*

$$s_k(\operatorname{ind} \xi) = \operatorname{tr}(s_k(\xi)). \tag{E3}$$

For $p = 2$ and k odd,

$$s_k(\text{ind } \xi) = \text{tr}(s_k(\xi)) + \tau_k,$$

where τ_k is of order at most 2.

Note. Whether or not $\tau_k = 0$ is easy to determine, but a little messy to describe.

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Department of Mathematics, Northwestern University, Evanston, Illinois 60201